

The Effect of Gravitational Tidal Forces on Vacuum Polarization: How to Undress a Photon

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ABSTRACT: The effect of gravitational tidal forces on photon propagation in curved spacetime is investigated. It is found that the imaginary part of the local refractive index $\text{Im } n(u; \omega)$ may be negative as well as positive, corresponding to a local amplification as well as attenuation of the amplitude of the renormalized photon field. This is interpreted in terms of the effect of tidal forces on the virtual e^+e^- cloud surrounding the bare photon field—a positive/negative $\text{Im } n(u; \omega)$ corresponds to an increased dressing/undressing of the bare photon. Below threshold decays of the photon to e^+e^- pairs can occur. Photon undressing in the vicinity of a black hole singularity is described as an example. These results are shown to be consistent with unitarity and the optical theorem in curved spacetime, which is derived here both in a local form and integrated over the photon trajectory.

1. Introduction

A consequence of vacuum polarization in QED in curved spacetime is that photons do not simply propagate along classical null geodesic trajectories. Instead, the spacetime acts as a medium with a non-trivial refractive index. In a series of papers [1–4], we have developed the theory of photon propagation in QED in curved spacetime and calculated the full frequency dependence of the refractive index in terms of a geometric quantity, the Van Vleck-Morette (VVM) determinant, which characterizes the null geodesic congruence around the classical trajectory. This has resolved the long-standing puzzle originating from the discovery of Drummond and Hathrell [5] that at low frequencies, the phase velocity may be superluminal. The geometry induces a novel analytic structure for the Green functions and refractive index which modifies the conventional Kramers-Kronig dispersion relation and removes the apparent paradox between low-frequency superluminal motion and causality [3].

An initially surprising result of this work was that in some backgrounds, the imaginary part of the refractive index may be negative. In a conventional optical medium, $\text{Im } n(\omega)$ is always positive, corresponding to scattering of photons from the beam and a reduction in the amplitude. A negative $\text{Im } n(\omega)$ implies gain, with energy pumped in from an external source. In quantum field theory, $\text{Im } n(\omega)$ is related via the optical theorem to the rate of production of e^+e^- pairs and therefore vanishes; even for a massive “photon” above the pair production threshold, $\text{Im } n(\omega)$ would be manifestly positive. The fact that $\text{Im } n(\omega)$ can be negative in curved spacetime, and can be non-vanishing and positive below the e^+e^- threshold, requires a careful explanation and interpretation. In particular, it involves a reappraisal of the optical theorem in curved spacetime.

In this paper, we summarize the resolution of this problem, leaving the full formal development to a longer paper [6]. We also provide an intuitive picture of how the amplitude may either increase or decrease depending on the sign of $\text{Im } n(\omega)$ based on the idea of “dressing” a renormalized quantum field.

In QED, vacuum polarization means we may regard the renormalized photon field as the bare field dressed with a cloud of virtual e^+e^- pairs. Since the electron has a mass, this cloud has a scale characterized by the electron Compton wavelength $\lambda_c = h/mc$, so the renormalized photon should be thought of as an extended body with size λ_c propagating through curved spacetime with scale $L \sim 1/\sqrt{\mathcal{R}}$, where \mathcal{R} is a typical curvature. Classically, massless point particles follow null geodesics in curved spacetime. Extended bodies, on the other hand, are subject to gravitational tidal forces because nearby geodesics can diverge or converge. In quantum field theory, because of vacuum polarization, we therefore expect that when L becomes

comparable to λ_c , these tidal forces will distort the virtual cloud and affect the degree of dressing of the renormalized photon.

The conceptual picture that we shall confirm is the following. To linear order in the curvature, when the rate of change of the acceleration of nearby geodesics is positive—in the direction of increased stretching—more virtual e^+e^- pairs are produced and the photon’s amplitude decreases. This is illustrated in the top diagram of Fig. 1 which indicates that the photon is becoming “more dressed”. If the rate of change of the acceleration is at least constant over a finite region along the geodesic then this can interpreted as real pair production and the amplitude reduces at a rate which is at least as great as exponential. On the contrary, when the rate of change of the acceleration of nearby geodesics is negative—in the direction of increased squeezing—virtual e^+e^- pairs are forced back into the photon. In this case the amplitude of is amplified. This is illustrated in the lower diagram of Fig. 1.

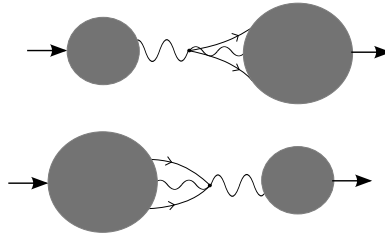


Figure 1: Diagrams which illustrate the the cases where the photon amplitude decreases (top) and increases (bottom) arising from the production of more/less virtual e^+e^- pairs, respectively. The top/bottom cases occurs when the rate of change of the acceleration on nearby geodesics is positive/negative.

This mechanism gives rise to a perturbative contribution to $\text{Im } n(\omega)$ of order $\omega \partial_u \mathcal{R}/m^4$ (where u is a lightcone coordinate along the null geodesic) in cases where the curvature is not constant along the photon trajectory. $\text{Im } n(\omega)$ can be positive or negative and the photon amplitude can decrease (even below threshold) or increase. In many respects, the situation resembles the transient phase immediately following an interaction being turned on in conventional flat spacetime QFT. In both cases, the possibility of amplification as well as attenuation originates in the loss of translation invariance, which enables energy conservation to be circumvented.

In addition to this very intuitive mechanism there is a further effect, non-perturbative in $\omega\sqrt{\mathcal{R}}/m^2$, which can give rise to an imaginary part for the refractive index. This is related to the occurrence of *conjugate points* along the photon’s null geodesic γ . Two points on γ which can be linked by a continuous family of geodesics arbitrarily close to γ are said to be conjugate. The existence of these nearby geodesics implies there are zero modes in the fluctuations around the classical path and conse-

quently singularities in the VVM matrix [1–3]. This is the ultimate geometric origin of the novel analytic structure for the refractive index in curved spacetime and in some cases may imply the existence of a non-vanishing contribution to $\text{Im } n(\omega)$. This effect is present even in the simple example of symmetric plane waves, or Cahen-Wallach spaces [7], where the curvature is constant along the photon path. In these cases, $\text{Im } n(\omega)$ is always positive,¹ a result which can now be understood from our analysis of the optical theorem in curved spacetime. For other backgrounds, there is no restriction on the sign of $\text{Im } n(\omega)$.

Clearly, the geometry of fluctuations around the classical photon trajectory is central to this analysis. Consider a null congruence centred on γ and let $z^i(u)$, $i = 1, 2$ be the two spacelike connecting vectors linking neighbouring geodesics. The $z^i(u)$ satisfy the Jacobi equation

$$\ddot{z}^i + h_{ij}(u)z^j = 0 \quad (1)$$

where $h_{ij}(u) = R^i_{uju}(u)$, the components of the Riemann tensor in a frame associated to the Brinkmann coordinates (u, v, z^i) adapted to the geodesic γ , which is the curve $v = z^i = 0$. The metric in the neighbourhood of γ takes the form

$$ds^2 = 2du dv - h_{ij}(u)z^i z^j du^2 + dz^i dz^i . \quad (2)$$

This metric is known as the Penrose limit for the given background and null geodesic γ and encodes the geometry of geodesic deviation [8, 9]. For later use, we also introduce the Rosen coordinates (u, V, x^a) , $a = 1, 2$, in which the approximate metric (2) takes the form

$$ds^2 = 2du dV + C_{ab}(u)dx^a dx^b . \quad (3)$$

The affine parameter u along γ is common to both sets of coordinates.

A key role is played by the Van Vleck-Morette determinant $\Delta(x, x')$ evaluated at the two points $x, x' \in \gamma$ with affine parameters u and u' , so we write $\Delta(u, u')$. There are different ways to define the VVM determinant but one very physical way is via solutions of the geodesic equation (1). If we write the solution for the Jacobi field $z^i(u)$ in terms of some initial data at u' ,

$$z^i(u) = \mathcal{B}^i_j(u, u')z^j(u') + \mathcal{A}^i_j(u, u')\dot{z}^j(u') , \quad (4)$$

then [3, 4]

$$\Delta(u, u') = \frac{(u - u')^2}{\det \mathcal{A}(u, u')} . \quad (5)$$

¹Note that in refs. [3, 4], there is an overall sign error in the refractive index for scalar QED. In particular, $\text{Im } n(\omega)$ for scalar QED in a Ricci-flat symmetric plane wave should be positive.

2. Vacuum Polarization and the Refractive Index

The Penrose limit will be a good approximation for a general curved spacetime provided the curvature scale is greater than the electron Compton wavelength, i.e. $L \gg \lambda_c$, or equivalently $\mathcal{R}/m^2 \ll 1$, and we work in the geometric optics, or eikonal, approximation where the curvature scale is large compared to the photon wavelength, i.e. $L \gg \lambda$, or $\mathcal{R}/\omega^2 \ll 1$. We refer to these two conditions as the “WKB limit”. This leaves the dimensionless combination $\omega\sqrt{\mathcal{R}}/m^2$ of the three scales as a free parameter in our calculation of the full frequency dependence of $n(\omega)$.

To simplify the presentation and focus on the key issues of the sign of $\text{Im } n(\omega)$ and the optical theorem, we quote results here for a model field theory comprising a massless scalar (photon) A interacting with two massive scalars (electrons) ϕ through an $eA\phi^2$ coupling. The case of QED involves some further subtleties with renormalization and is described elsewhere [6].

In the leading eikonal approximation, the solution of the free wave equation $\square A^{(0)}(x) = 0$ takes the approximate form

$$A^{(0)}(x) = g(x)^{-1/4} e^{i\omega\vartheta(x)} , \quad (6)$$

where $g = \det g_{\mu\nu}$ and $k^\mu = \partial^\mu \vartheta(x)$ is the tangent vector to a null congruence $g(k, k) = 0$. Along the geodesics $\vartheta(x) = V$ is constant while the affine parameter u varies. If we pick out γ to be the geodesic with $V = 0$ then in the neighbourhood of γ , V is identified with the Rosen coordinate.

In our model field theory, vacuum polarization is given by the usual QED-like one-loop graph of $\mathcal{O}(e^2)$ with the scalars ϕ in the loop and external A lines. Since the usual notion of a particle is inherently ambiguous in a general curved spacetime, we shall focus on the behaviour of the field itself and use the techniques of non-equilibrium QFT to set-up an initial value problem. We therefore imagine that on the null surface $u = u_0$ the coupling constant e is turned on instantaneously and then watch as the bare field evolves along the null direction u .² The field satisfies a non-local equation of the form

$$\square A(x) + e^2 \int d^4x' \sqrt{g(x')} \Pi_R(x, x') A(x') = 0 , \quad (7)$$

where $\Pi_R(x, x')$ is the one-loop retarded self-energy. Since the latter is only non-vanishing when x' lies in the past light-cone of x , the equation is manifestly causal.

²Strictly speaking a null surface is not a Cauchy surface. However, in the philosophy of light-cone quantization one can imagine defining boundary conditions in the transverse directions (V, x^a) which have the effect of giving a well-defined initial value problem.

The aim is to compute the solution of this perturbatively in e starting with the classical solution (6) in the form

$$A(x) = A^{(0)}(x) (1 + ie^2 \mathcal{Q}^{(1)}(x) + \dots) . \quad (8)$$

The one-loop retarded self-energy is

$$\Pi_R(x, x') = i\theta(u - u') \text{Im } G_F(x, x')^2 , \quad (9)$$

where $G_F(x, x')$ is the Feynman propagator of $\phi(x)$, and in order to extract the one-loop correction we need to calculate the integral

$$\int d^4x' \sqrt{g(x')} \Pi_R(x, x') A^{(0)}(x') . \quad (10)$$

This integral becomes tractable in the WKB limit, where we can replace a general background spacetime by its Penrose limit (2) or (3).

First, we introduce some notation which will be useful when we discuss the optical theorem. The general on-shell modes for a classical field of mass m in a plane wave background in Rosen coordinates are

$$\phi_p(x) = g(u)^{-1/4} \exp \left[i\omega V + ip_a x^a - \frac{i}{2\omega} (\psi^{ab}(u) p_a p_b + m^2 u) \right] , \quad (11)$$

where $p = (\omega, p_a)$ includes a transverse momentum and we have defined the 2×2 matrix

$$\psi(u) = \int^u du' C(u')^{-1} . \quad (12)$$

This matrix is related to the VVM determinant via

$$\Delta(u, u') = \frac{1}{\sqrt{g(u)g(u')}} \cdot \frac{(u - u')^2}{\det \psi(u, u')} . \quad (13)$$

These modes satisfy the orthonormality property,

$$(\phi_{p'}, \phi_p) \equiv i \int d^3x \sqrt{g} \phi_{p'}^*(x) \overleftrightarrow{\partial}_V \phi_p(x) = 2\omega(2\pi)^3 \delta(\omega - \omega') \delta^{(2)}(p_a - p'_a) , \quad (14)$$

where the integral is over $x = (V, x^a)$. The original massless photon modes $A^{(0)}$ of (6) were chosen with $p_a = 0$.

Now define the partial “Fourier transform” of the self-energy with respect to these (massless) modes:

$$\begin{aligned} & (2\pi)^3 \delta(\omega - \omega') \delta^{(2)}(p_a - p'_a) \tilde{\Pi}_R(u, u'; \omega, p_a) \\ &= \int d^3x \sqrt{g} \int d^3x' \sqrt{g'} \phi_p^*(x) \Pi_R(x, x') \phi_{p'}(x') . \end{aligned} \quad (15)$$

Solving the field equation (7), and taking into account that $\Pi_R(x, x')$ has support only for $u' < u$, we can then write the quantum correction $\mathcal{Q}^{(1)}$ of (8) in the form:

$$\mathcal{Q}^{(1)}(u) = \frac{1}{2\omega} \int_{u_0}^u du'' \int_{u_0}^{u''} du' \tilde{\Pi}_R(u'', u'; \omega, 0) . \quad (16)$$

Returning to (10), we now summarize some of the key points of the calculation, while the full details can be found in [6]:

(a) In the WKB limit, for a fixed x on the null geodesic γ , the integral over x' is dominated by a saddle point with $x' \in \gamma$ in the causal past of x . To leading order, the metric can be replaced by the approximate plane wave metric (2) or (3) which captures the fluctuations around γ to Gaussian order.

(b) The propagator for a massive scalar field in a plane wave is known exactly [10]:

$$G_F(x, x') = i\sqrt{\Delta(x, x')} \int_0^\infty \frac{dT}{(4\pi T)^2} e^{-im^2 T + \frac{i\sigma(x, x')}{2T}} . \quad (17)$$

where $\sigma(x, x')$ is the geodesic interval between x and x' and the saddle point over the x' , T_1 and T_2 (one for each propagator) integrals is at

$$V' = x'^a = 0 , \quad u' = u - 2m^2\omega \frac{T_1 T_2}{T_1 + T_2} . \quad (18)$$

What remains is an integral over the Feynman parameter $0 \leq \xi = \frac{T_1}{T_1 + T_2} \leq 1$ and the point x' on γ labelled by affine parameter u' with $u_0 \leq u' \leq u$. Performing the Gaussian integrals around the saddle-point gives the equation for the quantum correction $\mathcal{Q}^{(1)}(x)$. Evaluating this along γ gives

$$\partial_u \mathcal{Q}^{(1)}(u) = \frac{1}{32\pi^2\omega} \int_{u_0}^u du' \frac{\sqrt{\Delta(u, u')}}{u - u'} \int_0^1 d\xi e^{-\frac{im^2(u-u')}{2\omega\xi(1-\xi)}} , \quad (19)$$

which can easily be integrated with initial condition $\mathcal{Q}^{(1)}(u_0) = 0$ giving

$$\mathcal{Q}^{(1)}(u) = \frac{1}{32\pi^2\omega} \int_{u_0}^u du'' \int_{u_0}^{u''} du' \frac{\sqrt{\Delta(u'', u')}}{u'' - u'} \int_0^1 d\xi e^{-\frac{im^2(u''-u')}{2\omega\xi(1-\xi)}} . \quad (20)$$

The curvature is encoded in the Van Vleck-Morette determinant. The result (19) is divergent for two entirely different reasons: (i) there is the standard UV divergence as $u' \rightarrow u$. This is the same divergence as in flat space that is absorbed by a mass counterterm for the A field. This divergence does not affect the imaginary part of $\mathcal{Q}^{(1)}$ which is our main focus here, so we will not explain this point any further; (ii) there is a divergence whenever u' and u are conjugate points coming from the

VVM determinant. These divergences are treated by implementing the real space Feynman prescription, $u - u' \rightarrow u - u' - i0^+$.

Also note that because of the symmetry $\Delta(x, x') = \Delta(x', x)$, the imaginary part of $\mathcal{Q}^{(1)}(u)$ is given simply by extending the range of the integration from $u_0 \leq u' \leq u''$ to $u_0 \leq u' \leq u$. We find

$$\text{Im } \mathcal{Q}^{(1)}(u) = -\frac{i}{64\pi^2\omega} \int_{u_0}^u du'' \int_{u_0}^u du' \frac{\sqrt{\Delta(u'', u')}}{u'' - u'} \int_0^1 d\xi e^{-\frac{im^2(u''-u')}{2\omega\xi(1-\xi)}}. \quad (21)$$

This quantity appears in the optical theorem, as described in section 3.

(c) In general the one-loop correction can grow as a function of u and it becomes necessary to perform a re-summation using the dynamical renormalization group [11]. This results in an exponentiation of the result and the solution to the quantum corrected field equation (7) is then

$$A(u) = g(u)^{-1/4} e^{i\omega V} e^{ie^2 \mathcal{Q}^{(1)}(u)} \quad (22)$$

The result can be interpreted in terms of a position-dependent refractive index

$$n(u; \omega) = 1 + \frac{e^2}{\omega} \partial_u \mathcal{Q}^{(1)}(u) \quad (23)$$

with a real part that describes a non-trivial correction to the phase velocity of the massless field $A(x)$ and an imaginary part that describes a non-trivial correction to the amplitude

$$|A(u)| = |A(u_0)| \exp \left[-\omega \int_{u_0}^u du \text{Im } n(u; \omega) \right] \quad (24)$$

along γ .

The result (19) is already non-trivial even in flat space. This describes a transient region where the bare field A at $u = u_0$ renormalizes and becomes a dressed field in real time. Indeed, as u increases, $\text{Im } \partial_u \mathcal{Q}^{(1)}(u)$ settles down after some oscillations to a positive constant and the ratio $|A(u)/A(u_0)|$ approaches the wavefunction renormalization factor that one would calculate in the equilibrium theory:

$$\left| \frac{A(u)}{A(u_0)} \right| \rightarrow \exp \left[-\frac{e^2}{6\pi^2 m^2} \right] = 1 - \frac{e^2}{6\pi^2 m^2} + \dots \quad (25)$$

Notice that in $A\phi^2$ theory in $d = 4$, wavefunction renormalization is a finite effect. The decrease in the renormalized amplitude can be interpreted in terms of the intuitive picture of the bare field surrounded by a virtual cloud of ϕ pairs. As this dressing increases, the amplitude falls.

We can decouple this transient initial dressing of the field from the effective of gravitational tidal forces by taking $u_0 \rightarrow -\infty$ and then turning on the curvature at some finite u after the transients have died out. However, we must not forget the initial dressing process which is essential in maintaining consistency with unitarity, as explained later in our discussion of the optical theorem.

In the weak curvature limit $\mathcal{R} \ll m$, we can expand the VVM determinant to linear order in the curvature

$$\Delta(u, u') = 1 + \frac{1}{6} R_{uu}(u)(u - u')^2 - \frac{1}{12} \partial_u R_{uu}(u)(u - u')^3 + \dots \quad (26)$$

where the Ricci tensor $R_{uu} = R^i_{uiu}$. Substituting this expansion into the integral (19), and after UV regularization, we determine the correction to the refractive index

$$n(u; \omega) = 1 - \frac{e^2}{180\pi^2 m^4} R_{uu}(u) - \frac{ie^2 \omega}{420\pi^2 m^6} \partial_u R_{uu}(u) , \quad (27)$$

In this approximation, the quantum correction to the amplitude relative to a point $u' \gg u_0$ is

$$|A(u)| = |A(u')| \exp \left[\frac{e^2 \omega^2}{420\pi^2 m^6} (R_{uu}(u) - R_{uu}(u')) \right] . \quad (28)$$

If the Ricci curvature component $R_{uu}(u)$ decreases along γ , then from the Jacobi equation (1) we see that the rate of acceleration of the nearby geodesics is positive, *i.e* the tidal forces are increasing in the direction of stretching the virtual cloud. In this case, the amplitude decreases because more virtual pairs are being produced: A is becoming more dressed. On the contrary, if $R_{uu}(u)$ increases along γ , *i.e* the tidal forces are increasing in the direction of squeezing, then the amplitude increases because virtual pairs are interacting to produce more A : A is being “undressed” and is returning to its bare state. These effects are illustrated heuristically in Fig. 1.

We now come to the second, non-perturbative, mechanism which can produce an imaginary part for the refractive index. If $R_{uu}(u)$ is constant, in which case the approximate metric (2) and (3) is a symmetric plane wave, or Cahen-Wallach space [7], then to linear order in the curvature the amplitude is constant. In fact, $\text{Im } n(\omega) = 0$ to all orders in the curvature expansion. However, in the case where at least one of the eigenvalues of the constant h_{ij} is negative there is a positive contribution to $\text{Im } n(\omega)$, so a decaying amplitude, which is non-perturbative in the curvature. The simplest case to consider is $h_{ij} = -\sigma^2 \delta_{ij}$. In this case, the VVM determinant is

$$\Delta(u, u') = \left[\frac{\sigma(u - u')}{\sinh \sigma(u - u')} \right]^2 . \quad (29)$$

Introducing $t = u - u'$ we have

$$n(\omega) = 1 + \frac{e^2}{32\pi^2\omega^2} \int_0^\infty \frac{dt}{\sinh(\sigma t)} \int_0^1 d\xi e^{-\frac{im^2 t}{2\omega\xi(1-\xi)}} . \quad (30)$$

$\text{Im } n(\omega)$ can be computed by rotating the $t = u - u'$ integral $t \rightarrow -it + i0^+$ and picking up contributions from the poles at $t = \frac{n\pi}{\sigma}$, $n = 1, 2, \dots$. This gives

$$\text{Im } n(\omega) = \frac{e^2}{16\pi\omega^2} \int_0^1 d\xi \frac{1}{1 + e^{\frac{\pi m^2}{2\omega\sigma\xi(1-\xi)}}} , \quad (31)$$

which is, indeed, non-perturbative in the curvature. In this case the result is independent of u and so it implies a constant rate of production of ϕ pairs. Such a process is kinematically disallowed in flat space, but there is no such threshold constraint in curved space.

This result leads to a very non-trivial check of the formalism in the following way [6]. If we consider the space $dS_3 \times R$ with metric

$$ds^2 = -dt^2 + \cosh^2(\alpha t) (d\theta^2 + \sin^2 \theta d\phi^2) + d\chi^2 , \quad (32)$$

then the plane wave limit about the geodesic $t = \chi$ and a fixed point on the S^2 , *e.g.* $\theta = \frac{\pi}{2}$, $\phi = 0$, is obtained by transforming

$$t = \frac{1}{\sqrt{2}}(u - v) , \quad \chi = \frac{1}{\sqrt{2}}(u + v) , \quad (33)$$

and then expanding in powers of v and $x^1 = \theta - \frac{\pi}{2}$, $x^2 = \phi$:

$$ds^2 \rightarrow 2du dv + \cosh^2\left(\frac{\alpha u}{\sqrt{2}}\right) dx^a dx^a . \quad (34)$$

This is a symmetric plane wave in Rosen coordinates (3) with $h_{ij} = -\frac{\alpha^2}{2}\delta_{ij}$. The non-trivial check on (31) comes about because the decays of scalar fields $A \rightarrow \phi\phi$ can be calculated exactly in de Sitter space due to large amount of symmetry as a tree-level process for any masses. In our case A is massless and using the results in [12,13] and taking the limits $\omega \gg \alpha$ and $m \gg \alpha$ one gets a decay rate which is precisely $2\omega \text{Im } n(\omega)$ as found in (31).

We therefore see that in curved spacetime, the effect of gravitational tidal forces on the virtual cloud surrounding the bare field means that the photon can become increasingly “undressed” as well as dressed as it propagates through spacetime. An especially interesting example of this gravitational undressing phenomenon is photon propagation near a singularity, for example a black hole. The Penrose limit near the singularity has the universal form of a homogeneous plane wave [9]

$$h_{ij}(u) = \frac{1 - \alpha_i^2}{4u^2} \delta_{ij} , \quad (35)$$

with α_i being either real or imaginary (and choosing $\alpha_i \geq 0$ in the former). For example, the near-singularity Penrose limit of a null geodesic with non-vanishing angular momentum in the Schwarzschild solution is described by such a plane wave with $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{7}{5}$. As the singularity is approached $u \rightarrow 0$ (with $u < 0$) the u' integral is dominated by its endpoint $u' = u_0 = -\infty$ and we find

$$\text{Im } \mathcal{Q}^{(1)}(u) \longrightarrow -|u|^{\frac{\alpha_1 + \alpha_2 + 2}{4}}. \quad (36)$$

The amplitude therefore increases and the photon becomes progressively undressed as the singularity is approached. Notice that the amplitude remains finite even though $\text{Im } n(u; \omega)$ itself diverges if $\alpha_1 + \alpha_2 < 2$ as in the case of Schwarzschild. When either of the α_i is imaginary the amplitude has a oscillating behaviour.

3. Optical Theorem

The final issue we need to address is how the optical theorem and unitarity can be satisfied if the amplitude can increase and $\text{Im } n(u, \omega)$ can be negative. The answer to this puzzle is that we have only shown that the amplitude can increase locally in a region well away from the initial-value surface. In order to verify that unitarity is satisfied, and that the decrease in the amplitude is related to the probability of the production of ϕ pairs, we have to consider the amplitude over the whole region from the initial-value surface u_0 to u . It turns out that while the amplitude can increase locally, the overall amplitude as measured from u_0 always decreases. The initial transient dressing of the A field plays an important rôle since it is only a field that is already dressed that can increase its amplitude by becoming more undressed; a bare field, however, cannot be undressed any further!

To derive the optical theorem, note first that

$$\int_{u_0}^u du'' \text{Im } n(u''; \omega) = \frac{e^2}{2\omega^2} \int_{u_0}^u du'' \int_{u_0}^{u''} du' \text{Im } \tilde{\Pi}_R(u'', u'; \omega, 0) = \frac{e^2}{\omega} \text{Im } \mathcal{Q}^{(1)}(u). \quad (37)$$

where $\text{Im } \mathcal{Q}^{(1)}(u)$ is given in eq.(21). Now consider the the transition probability for the tree-level process $A \rightarrow \phi\phi$:

$$P_{A \rightarrow \phi\phi}(u) = \frac{e^2}{\mathcal{N}} \int \frac{d^3 p_1}{(2\pi)^3 2\omega_1} \frac{d^3 p_2}{(2\pi)^3 2\omega_2} \left| \int_{u_0}^u d^4 x \sqrt{g} A^{(0)}(x)^\dagger \phi_{p_1}(x) \phi_{p_2}(x) \right|^2. \quad (38)$$

Here, $\phi_p(x)$ are the massive on-shell modes defined in (11) and the integrals are over the three-dimensional $p = (\omega, p_a)$. The normalization factor $\mathcal{N} = (A^{(0)}, A^{(0)}) = 2\omega\delta^{(3)}(0)$ cancels against an overall integration on (V, x^a) . Recall that the Wightman

functions $G_+(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$ and $G_-(x, x') = \langle 0 | \phi(x') \phi(x) | 0 \rangle$ are given in terms of these modes by

$$G_{\pm}(x, x') = \pm \int_{\omega \gtrless 0} \frac{d^3 p}{(2\pi)^3 2\omega} \phi_p(x) \phi_p(x')^\dagger . \quad (39)$$

and that the frequencies are all positive since they refer to physical particles in (38). The retarded self-energy (9) is expressed in terms of the Wightman functions by

$$\Pi_R(x, x') = i\theta(u - u') [G_+(x, x')^2 - G_-(x, x')^2] . \quad (40)$$

A short calculation using the properties $G_+(x, x')^* = G_-(x, x')$ and $G_+(x', x) = G_-(x, x')$, and noting that $\int dV' G_-(x, x')^2 A^{(0)}(x') = 0$ for positive ω , now shows that

$$P_{A \rightarrow \phi\phi}(u) = \frac{e^2}{\omega} \int_{u_0}^u du'' \int_{u_0}^{u''} du' \operatorname{Im} \tilde{\Pi}_R(u'', u'; \omega, 0) , \quad (41)$$

and so

$$P_{A \rightarrow \phi\phi}(u) = 2\omega \int_{u_0}^u du'' \operatorname{Im} n(u''; \omega) . \quad (42)$$

This is the statement of the optical theorem in curved spacetime. Unitarity is respected and the integral over the whole trajectory of $\operatorname{Im} n(u; \omega)$ is positive. This shows how unitarity prevents a local amplification of the amplitude from becoming unbounded.

We can also write a local version, defining $\Gamma(u) = \partial_u P_{A \rightarrow \phi\phi}(u)$ as the “instantaneous rate” of pair production. Then,

$$\Gamma(u) = 2\omega \operatorname{Im} n(u; \omega) = \frac{e^2}{\omega} \int_{u_0}^u du' \operatorname{Im} \tilde{\Pi}_R(u, u'; \omega, 0) . \quad (43)$$

Unlike the integrated form, there is no positivity constraint on (43) in general. This explains why there is no conflict with unitarity in the examples where we have found $\operatorname{Im} n(u; \omega) < 0$. However, in cases where we have translation invariance along the photon trajectory, as in flat spacetime, $n(\omega)$ is u -independent—as long as one is beyond the transient region—and $P_{A \rightarrow \phi\phi}(u)$ is proportional to u , so unitarity implies $\operatorname{Im} n(\omega) > 0$ and $\Gamma > 0$ can be properly interpreted as the rate of $A \rightarrow \phi\phi$.

4. Conclusions

In conclusion, we have shown how the effect of gravitational tidal forces on vacuum polarization can alter the dressing of a photon as it propagates through

space. In particular, we have seen how the imaginary part $\text{Im } n(u; \omega)$ of the position-dependent refractive index can be negative as well as positive, corresponding to “undressing” of the photon rather than the conventional dressing. Two mechanisms were identified. The first, of order $\omega \partial_u \mathcal{R}/m^4$, admits an intuitive interpretation in terms of curvature variations along the photon’s trajectory altering the balance of the bare field with its virtual e^+e^- cloud, with increasing stretching (squeezing) giving rise to more dressing (undressing). The second, which is non-perturbative in $\omega \mathcal{R}/m^2$, can occur even when the curvature is constant and is related to the existence of conjugate points on the photon’s null geodesic.

Nevertheless, unitarity is still respected and the optical theorem still holds in curved spacetime. In its integrated form (41), (42), it relates the total probability for e^+e^- pair production to the integral of $\text{Im } n(u; \omega)$ along the photon trajectory, which is manifestly positive. Except in the special case of translation invariance along the null geodesic, the corresponding local form (43) has no positivity constraint and the usual interpretation of $\text{Im } n(u; \omega)$ as the rate of pair production can break down. However, it does describe the variation of the amplitude and its interpretation in terms of dressing and undressing of the photon field. In a sense, photon propagation in curved spacetime resembles the initial transient phase in flat spacetime, with the characteristic features of an oscillating amplitude and below-threshold decay.

In this paper, we have described our results in terms of a scalar $A\phi^2$ model of QED. In QED itself, there are two further complications—the photon has a polarization, whose direction influences the dynamical dressing, and there are subtleties related to renormalization, requiring a reformulation of the initial-value problem to circumvent the short-distance divergences at the initial value surface caused by the lack of asymptotic freedom of QED. These issues are explored in a more comprehensive treatment of the effect of gravitational tidal forces on renormalized quantum fields in ref. [6].

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